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Decay of Correlations in One and Two Dimensions

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An exposition of some methods of proving exponential (stretched exponential) decay of correlations is given. One-dimensional strictly hyperbolic and quadratic maps and two-dimensional piecewise smooth, uniformly hyperbolic maps are considered. The emphasis is on the fundamental constructions of the Markov sieve method due to Bunimovich–Chernov–Sinai and those of Liverani's Hilbert metric method.

KEY WORDS: Correlation decay; smooth and piecewise smooth hyperbolic maps; Markov sieve; cone of functions; Hilbert metric.

1. INTRODUCTION

The main objective of this paper is to outline the different techniques of proving exponential (or stretched exponential) decay of correlations of continuous observables under the action of the iterates of a map of an interval or some two-dimensional (not necessarily connected) domains. The general setup looks as follows: An "observable" means a real-valued function $f(x)$ on the phase space $\mathscr{M} \ni x$ which at least should be a topological space, in order to speak about continuous functions, but in fact we assume that $\mathscr{M}$ is a Riemannian manifold. There is given a continuous or piecewise continuous map T on $\mathscr{M}$, having an invariant probability measure μ; let E denote the expectation with respect to μ. We investigate the question of whether the rate of the convergence

$$|\mathrm{E}f(x)f(T^n x) - (\mathrm{E}f(x))^2| \to 0$$

is exponential (or subexponential). The rate in principle depends both on f and on T, but usually we consider Hölder continuous f; then the question concerns only the nature of T.

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This latter statement is one of the reasons allowing the proof of the central limit theorem:

$$\mu\left\{\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(f(T^n x)-\mathrm{E}f(x)\right)<t\right\}\to\Phi_\sigma(t)$$

where Φ_σ is the integrated Gaussian density with zero mean and $0<\sigma<\infty$ standard deviation.

However, a much deeper fact lies behind the central limit theorem, namely that the multiple correlation functions

$$\mathrm{E}f(x)\,f(T^{n_1}x)\,f(T^{n_2}x)\cdots f(T^{n_k}x)$$

tend to zero sufficiently fast if $\mathrm{E}f(x)=0$ and n_j-n_{j-1} $(n_0=0)$ tend uniformly to ∞.

We focus our attention on smooth or piecewise smooth hyperbolic maps of the interval, or a two-dimensional Riemannian surface having invariant measure absolutely continuous with respect to the Lebesgue or Riemannian measure. The interesting cases are, however, those for which either the hyperbolicity or the smoothness is violated. A typical example for the first case is the famous quadratic map of the interval, while the billiard map is the most challenging example for strictly hyperbolic maps among several cylinders with singularities.

In the majority of cases instead of the map T itself one considers the Perron–Frobenius operator $\tilde{T}$ acting on measures defined on $\mathscr{M}$. In the most general case the Perron–Frobenius operator can be defined as follows:

Let T be a measurable map (not necessarily one-to-one) of a metric space $\mathscr{M}$ endowed with a finite measure μ, usually not invariant under the action of T. The Perron–Frobenius operator $\tilde{T}$ acts on the "space" of measures; it describes the evolution of the initial measure μ under the action of T by the definition

$$\forall\ \text{Borel measurable } A\subset\mathscr{M}:\qquad \tilde{T}\mu(A)=\mu(T^{-1}A)$$

Recall that for usual Markov chains with countable states the decay of correlations and the rate of convergence of some initial distribution to an invariant one can be derived simultaneously from the spectral properties of the probability transition matrix whose analog is the Perron–Frobenius operator.

Pollicott[19] and Ruelle[20] developed an effective method of computing the Fourier transform of the correlation function (in fact the spectrum of the Perron–Frobenius operator) using the Fredholm determinant of the map. The general Fredholm theory was worked out by Grothendieck.[13] The theory of Pollicott and Ruelle is based on the thermodynamic formalism applied to dynamical systems—invented by R. Bowen, D. Ruelle, and Ya. G. Sinai—which is beyond the scope of this paper.

There is an abundant literature on the numerical estimation of the rate of correlation decay; we refer to the recent paper of Garrido and Gallavotti.[12]

Our purpose is more modest: to expose the Markov partition method of Sinai's school, and the method of the Hilbert metric due to C. Liverani. Both of them are attempts toward the proof of exponential correlation decay for billiards.

The structure of the paper is as follows: in Section 2 we cite the recent result of L.-S. Young on one-dimensional quadratic map. Section 3 summarizes the theory of two-dimensional smooth (or smooth with singularities) hyperbolic maps. This section contains the common ingredients of the method of the Markov sieve and the method of the Hilbert metric treated in Sections 4 and 5, respectively.

2. QUADRATIC MAP OF INTERVAL

Here we cite a recent result of Young[27] concerning the quadratic map, for two reasons: (i) it is the simplest example where exponential correlation decay is established and the uniform hyperbolicity is violated; (ii) Liverani's ideas will be demonstrated on one-dimensional expanding maps.

Consider the map f_a of $[-1, 1]$ into itself defined by $f_a(x) = 1 - ax^2$. The map $f_a(x)$ is not everywhere expanding; therefore the well-known theorem of Yakobson[26,4] stating that there is a positive Lebesgue measure set Δ in the parameter set such that if $a \in \Delta$, then f_a has an absolutely continuous invariant measure μ was a surprising achievement. Benedicks and Carleson[5] proved that for a positive measure subset $\Delta' \subset \Delta$ the density function of the invariant measure is a sum of a function with bounded variation and a function having inverse square-root singularities. In this case the absolutely continuous invariant measure is unique, and the natural extension of (f_a, μ) is equivalent to the Bernoulli shift.

Young's results concern the class of f_a of the above type and state that if $\varphi\colon [-1, 1] \to \mathbf{R}$ has bounded variation, then there exists a constant $0 < \lambda < 1$ depending only on a and another constant C depending on φ as well, such that

$$|\mathrm{E}\varphi(x)\,\varphi(f^n x) - (\mathrm{E}\varphi(x))^2| < C\lambda^n$$

Both the proof of the existence and unicity of μ and that of the decay of correlations are based on the investigation of the spectral properties of the Perron–Frobenius opertor $\tilde{f}$ applied to the density φ of a measure:

$$\tilde{f}\varphi(y) = \sum_{x \in f^{-1}y} \varphi(f^{-1}y)\,|D_x f^{-1}x| \tag{1}$$

The difficulty caused by the nonuniform expanding can be overcome by constructing a partition of $[-1, 1]$ into countable many intervals, shrinking exponentially around the critical point 0. For each interval I there exists a power p_I such that the iterates $f_a^{p_I}$ are uniformly expanding. The subtlety of this idea is to find the appropriate conditions which hold for a positive set of parameter values a and lead to the desired results.

3. GENERAL INFORMATION ON TWO-DIMENSIONAL HYPERBOLIC SYSTEMS

Definition 3.1 [Two-dimensional uniformly hyperbolic (Anosov) map]. Let $\mathscr{M}$ be a compact, smooth, connected, orientable Riemannian surface with tangent bundle $\mathscr{T}_x$ and let $T: \mathscr{M} \to \mathscr{M}$ be a C^2-diffeomorphism. Assume that for every $x \in \mathscr{M}$ there exist two vectors $v^s(x)$ and $v^u(x)$ depending continuously on x, called, respectively, stable and unstable vectors at x such that

$$D_x T v^s(x) = \lambda(x)\, v^s(x)$$

$$D_x T v^u(x) = \lambda'(x)\, v^u(x)$$

and there exist two positive constants $0 < \lambda < \mu < 1$ such that for every $x \in \mathscr{M}$, $\lambda \leqslant |\lambda(x)| \leqslant \mu$ and $\mu^{-1} \leqslant |\lambda'(x)| \leqslant \lambda^{-1}$.

Further we assume that the angle between the tangent vectors $v^s(x)$ and $v^u(x)$ is uniformly bounded from below (uniform transversality).

For the sake of definiteness we assume that $\mathscr{M}$ is homeomorphic to the 2-torus.

The simplest example for Anosov maps is the algebraic automorphism of the 2-torus:

Definition 3.2. Let $\mathscr{M} := \{(x, y) \mid 0 \leqslant x, y < 1\}$ be a 2-torus and T_0 be defined by $T_0(x, y) = (ax + by,\ cx + dy) \bmod(1)$, where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integer entries $\det A = \pm 1$, and both eigenvalues of A are real and $\neq \pm 1$. Obviously for every $x \in \mathcal{M}$, $v^s(x)$ $[v^u(x)]$ is tangent vector parallel to the eigenvector v^s (v^u) belonging to the eigenvalue $|\lambda| < 1$ $(|\lambda^{-1}|)$.

Having the prototype of the two-dimensional maps considered in this paper before defining the other ones, we briefly recapitulate the standard constructions and facts needed to study the ergodic properties of two-dimensional hyperbolic maps.

3.1. Invariant Stable (Unstable) Fibers

Proposition 3.3. For all $x \in \mathcal{M}$ there exist two C^1 smooth curves γ_x^s and γ_x^u tangent to $v^s(x)$ and $v^u(x)$ passing through x tangent to $v^s(x)$ and $v^u(x)$, respectively, such that $T\gamma_x^s \subset \gamma_{Tx}^s$ and $T^{-1}\gamma_x^u \subset \gamma_{T^{-1}x}^u$.[3]

The stable (unstable) fibers of an Anosov map are everywhere dense in $\mathcal{M}$ (transitivity); for our purposes it is more convenient to work with local fibers of length ε, and the choice of ε depends on the Riemannian metric only: $\gamma_x^{s,\mathrm{loc}}$ $(\gamma_x^{u,\mathrm{loc}})$ is a subinterval on γ_x^s (γ_x^u) of arc length (measured by the Riemannian metric) 2ε with center x. If ε is small enough, then for every pair $x, y \in \mathcal{M}$ there exists at most one $z \in \mathcal{M}$ such that $z \in \gamma_x^{s,\mathrm{loc}} \cap \gamma_y^{u,\mathrm{loc}}$.

Define $[x, y]$ by setting $z := [x, y]$ if z exists. If x and y are sufficiently close to each other (measured by the arc length of the geodesic connecting them), then the four points of $\mathcal{M}$: x, $[x, y]$, y, $[y, x]$ define a quadrilateral Q whose sides are local stable and unstable fibers; see Fig. 1. (In the sequel, hoping not to cause confusion, we omit the superscript "loc" in the notation of local fibers). There are two canonical isomorphisms

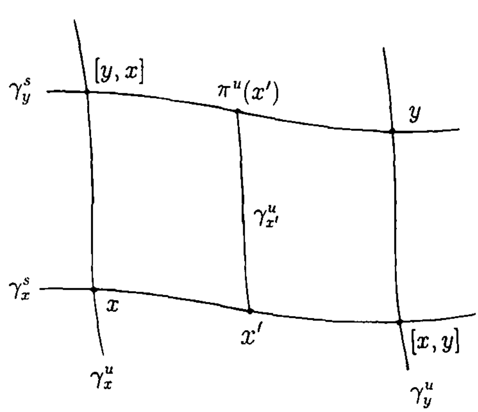

Fig. 1. Local stable and unstable manifolds. Canonical isomorphisms.

between the opposite sides of Q, namely the projection along the *local* stable (unstable) fibers:

$$x' \in \gamma_x^s \to \pi^u(x') := \gamma_{x'}^u \cap \gamma_{[y,x]}^s$$

if x' is between x and $[x, y]$ on γ_x^s and

$$x'' \in \gamma_x^u \to \pi^s(x'') := \gamma_{x''}^s \cap \gamma_{[x,y]}^u$$

if x'' is between x and $[y, x]$ on γ_x^u.

Notice that in the simplest case (algebraic automorphism of the 2-torus) the stable (unstable) fibers of T_0 are infinite straight lines (dense in $\mathscr{M}$) parallel to the eigenvector v^s (v^u).

Definition 3.4. A subset $Q \subset \mathscr{M}$ is called a *parallelogram* iff for every $x, y \in Q$, $[x, y] \in Q$. Set $\gamma_{x,Q}^s := \gamma_x^s \cap Q$ and $\gamma_{x,Q}^u := \gamma_x^u \cap Q$.

Remark. If T is an Anosov map, then the quadrilateral Q defined by x, $[x, y]$, y, $[y, x]$ is a parallelogram. We shall call such a parallelogram "full." The pieces of local fibers x, $[x, y]$ and $[y, x]$, y are called s-sides of Q, while x, $[y, x]$ and $[x, y]$, y are called u-sides of Q.

3.2. Invariant Measure

The above definition has only topological character. The existence of an absolutely continuous—with respect to the Riemannian volume—invariant probability measure should be postulated. To clarify the development of ideas, we recall that if such a measure exists, then it coincides with that constructed by Sinai[22] using Markov partitions (Bowen–Ruelle–Sinai measure).

On the other hand, the algebraic automorphism of the 2-torus preserves the Lebesgue measure.

In a recent paper Pesin[18] constructed a measure analogous to the Bowen–Ruelle–Sinai measure for dynamical systems with generalized hyperbolic attractors, like Lorentz-type and Lozi attractors. All the ingredients of the proof of stretched exponential correlation decay are established there.

Given an invariant measure for T, we can formulate the ergodic properties of uniform hyperbolic maps:

Proposition 3.5. Uniform hyperbolic maps are ergodic with respect to μ, obey the K-property (Kolmogorov mixing[22,23], and are equivalent to the Bernoulli shift.[7]

The most effective technique to prove ergodicity (more precisely, the fact that the phase space is a countable union of positive ergodic components) is Hopf's method, based on the absolute continuity (with respect to the arc length measure) of canonical isomorphisms π^u and π^s.[2,21]

3.3. Two-Dimensional Uniformly Hyperbolic Map with Singularities

Definition 3.6 (*UHS map*). Let $\mathcal{M}$ be the union of a finite number of compact, smooth, orientable Riemannian manifolds with C^2-smooth boundaries (in the course of this paper $\mathcal{M}$ is considered to be the union of a finite number of squares or cylinders). Assume that $\mathcal{M}$ is partitioned in two ways into unions of equal numbers of boxes,

$$\mathcal{M} = \mathcal{M}_1^+ \cup \cdots \cup \mathcal{M}_m^+ = \mathcal{M}_1^- \cup \cdots \cup \mathcal{M}_m^-$$

Two boxes of one partition can overlap at most on their boundaries, i.e.,

$$\mathcal{M}_i^\pm \cap \mathcal{M}_j^\pm = \partial\mathcal{M}_i^\pm \cap \partial\mathcal{M}_j^\pm$$

The map T is defined separately on each box $\mathcal{M}_i^+$, $i = 1,..., m$. It is an Anosov map in the sense of Definition 3.1 of the interior of each $\mathcal{M}_i^+$ onto the interior $\mathcal{M}_i^-$, $i = 1,..., m$.

For more details concerning UHS maps we refer to the classical monograph of Katok and Strelcyn[14]; for recent results we suggest the paper of Liverani and Wojtkowski.[17]

Chernov's examples—piecewise linear hyperbolic automorphisms (PLH; for definition see below) belong to this class of maps, while for the Poincaré section of the Sinai billiard flow with finite horizon the inequality $\lambda(x) \leqslant \mu$ fails: if x tends to the boundary $\partial\mathcal{M}_i^+$ then $\lambda(x) \to \infty$.

Formally T is not well defined on the set of points which belong to the boundaries of several plus-boxes: it has several values. We adopt the convention that the image of a subset of $\mathcal{M}$ under T contains all such values.

Let us introduce the singularity sets $\mathcal{S}^+$ and $\mathcal{S}^-$:

$$\mathcal{S}^\pm = \{x \in \mathcal{M} \mid x \text{ belongs to at least two of the boxes } \mathcal{M}_i^\pm,\ i = 1,..., m\}$$

The plus-singularity set $\mathcal{S}^+$ is a closed subset, and T is continuous on its complement. Similarly T^{-1} is continuous on the complement of $\mathcal{S}^-$.

Definition 3.7 (*PLH map*). Let T_0 be a hyperbolic algebraic automorphism of a 2-torus $\mathcal{M}$. Let us cut the torus $\mathcal{M}$ along several compact curves which are either closed or have common endpoints. These curves divide the torus into several pieces. We assume that these pieces can be shifted or rearranged in such a way that they fully cover the torus $\mathcal{M}$ again. As a result we get a piecewise linear transformation of $\mathcal{M}$.

For example, if T_0 is the map defined by the matrix

$$\begin{pmatrix} 1 & a \\ 1 & 1+a \end{pmatrix}$$

with real a, then $T = T_0^2$ is a piecewise linear transformation of $\mathcal{M}$ consisting of a finite (countable) number of continuous pieces if a is rational (irrational).

Denote by $\mathcal{S}$ the set of the above cut curves, and for every integer n set $\mathcal{S}_n = T^n \mathcal{S} : T^n$ (T^{-n}) are undefined and discontinuous on $\mathcal{S}_{-n+1}$ $(\mathcal{S}_{n-1})$. Set $\mathcal{S}_{n,m} = \mathcal{S}_n \cup \cdots \cup \mathcal{S}_m$. Now, T and its (positive and negative) iterates are defined on $\mathcal{M}_0 = \mathcal{M} \backslash \mathcal{S}_{-\infty,\infty}$ and they preserve the Lebesgue measure.

3.4. Billiard

For the sake of simplicity, instead of a comprehensive description of the billiard systems we give the definition of the simplest billiard system with finite horizon on the 2-torus.

Definition 3.8 (*Billiard map*). The configuration space of the billiard flow can be obtained from a 2-torus by discarding a finite number of open convex domains S_i (scatterers) with smooth boundaries. Assume that the scatterers are placed on the torus in such a way that no infinite straight line could be drawn on the configuration space (see Fig. 2a). A point particle is moving with unit velocity in te interior of the configuration space, and it reflects at the boundary according to the law: the angle of reflection is equal to the angle of incidence. This motion T_t is called billiard flow. T_t preserves the product of Lebesgue measure on the configuration space and the Haar measure of the unit circle of possible velocities. (Notice that on the boundary, only a half-circle of velocities is admissible.) The billiard map T is the discretization (a natural Poincaré section) of T_t: T brings an outgoing vector "just before reflection" into an outgoing vector "just after the next reflection." T is well defined for all outgoing vectors except the tangential ones to the scatterers. The phase space $\mathcal{M}$ of T consists of several cylinders (the boundaries ∂S_i of the scatterers $\times (-\pi/2, \pi/2)$; the angle φ is measured from the outer normal of the scatterer; see Fig. 2b).

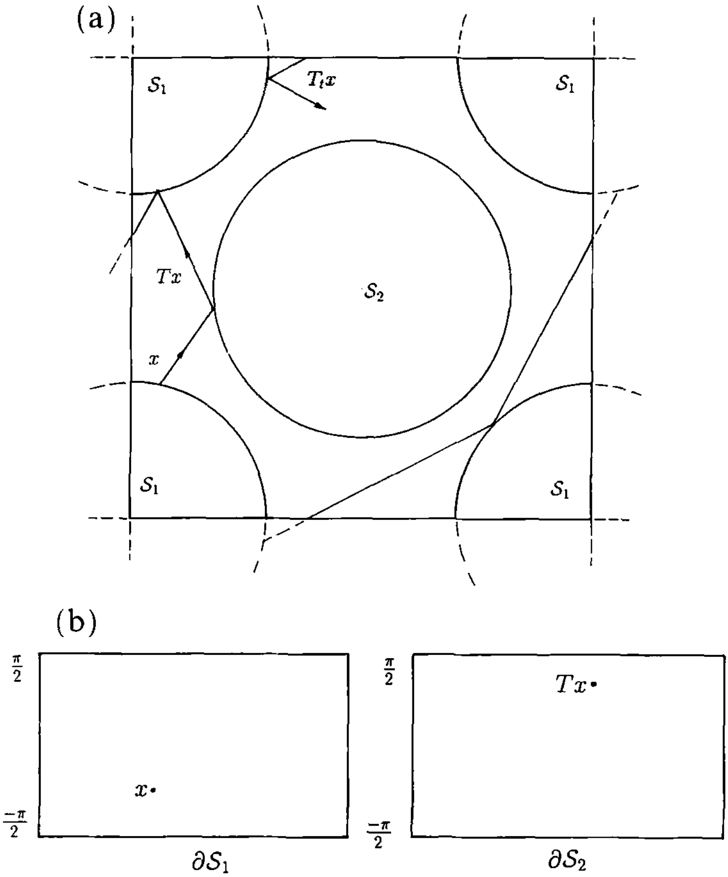

Fig. 2. (a) The billiard flow. (b) The billiard map.

Denote by $\mathscr{S}$ the union of the boundaries ($\varphi = \pm\pi/2$) of cylinders, and for every integer n set $\mathscr{S}_n = T^n\mathscr{S}$: T^n (T^{-n}) are undefined and discontinuous on $\mathscr{S}_{-n+1}$ ($\mathscr{S}_{n-1}$). Set $\mathscr{S}_{n,m} = \mathscr{S}_n \cup \cdots \cup \mathscr{S}_m$. Now, T and its (positive and negative) iterates are defined on $\mathscr{M}_0 = \mathscr{M} \backslash \mathscr{S}_{-\infty,\infty}$ and they preserve the absolutely continuous measure with respect to the Lebesgue measure, having density function equal to const $\times \cos\varphi$.

Proposition 3.9. The billiard map is ergodic, obseys the K-property[24] (Kolmogorov mixing), and is equivalent to the Bernoulli shift.[11]

Remark. All the singular hyperbolic systems considered in this paper obey the following property: the maximal number $M(n)$ of smooth lines belonging to $\bigcup_{i=-n}^{n} T^i \mathscr{S}^-$ that intersect at one point has polynomial growth. In fact for Liverani's proof it will be sufficient that in the case of the UHS map there exists an appropriate exponent $0<\zeta<1$ and a natural number n_0 such that

$$M(n_0)\lambda^{-\zeta n_0}<\text{const} \tag{2}$$

One of the main difficulties arises from the fact that Proposition 3.3 does not hold for singular systems; however, for almost every point $x \in \mathscr{M}$ there are local stable and unstable fibers (γ_x^s and γ_x^u) passing through it, but no *a priori* lower bound for their lengths exists. Let us denote by $r^s(x)$ $[r^u(x)]$ the length of γ_x^s (γ_x^u). The following proposition holds:

Proposition 3.10. There are two positive constants C and $\alpha \leqslant 1$ such that $\mu\{x: r^{s,u}(x)<\varepsilon\} \leqslant C\varepsilon^{\alpha}$.

Notice that for the piecewise linear hyperbolic automorphisms $\alpha=1$.

Due to Proposition 3.10 the canonical projections π^u and π^s can be defined for pairs of close stable (unstable) local fibers, say γ_x^s and γ_y^s (γ_x^u and γ_y^u) if x and y are sufficiently close and $r^s(x)$ and $r^s(y)$ $[r^u(x)$ and $r^u(y)]$ are sufficiently large. The canonical projections will be defined only for a positive subset (with respect to the arc length measure) of γ_x^s (γ_x^u). Nonetheless, they are absolutely continuous, which ensures that the positive ergodic components of T are of full measure and that T has the K-property and is equivalent to the Bernoulli shift on each such component. This last statement is true for the billiard map, too.

Observe that Definition 3.4 is meaningful also in the case of the UHS map. A parallelogram usually will be a two dimensional Cantor set in this case.

A fiber γ^s (γ^u) intersects a square G properly iff $\gamma^s \cap G$ ($\gamma^u \cap G$) is an inner subinterval of γ^s (γ^u).

Definition 3.11. Let Q be a quadrilateral whose boundary consists of four local fibers; then $A(Q)$ consisting of the intersections of local stable and unstable fibers properly intersecting Q is called the *maximal parallelogram inscribed in Q*.

In order to prove that a UHS map T has only one ergodic component, the so-called Sinai fundamental lemma[25,15]—stating, roughly speaking, that $Q \subset \mathscr{M}$ can be covered by sufficiently small squares the overwhelming majority of which contain an abundance of stable fibers—is inevitable.

Sinai's Lemma. A square G is called α-connecting iff the measure of the maximal parallelogram inscribed into G is at least α times the measure of G. For each square Q of diameter δ_0, $t>1$, and $\varepsilon>0$ there exist

$\alpha_Q < 1$, $\delta_Q \leqslant \delta_0$, and a natural m_Q such that, for each $\alpha \leqslant \alpha_Q$, $m \geqslant m_Q$, and $\delta_1 \leqslant \delta_Q$ we can construct a covering $\mathscr{G}(Q, \delta_1, m)$ made of squares of size $t\delta_1$ with the following properties:

(i) The number of squares G for which $\bigcup_{|n| \leqslant m_Q} T^n \mathscr{S} \cap G = \varnothing$ and G is not α-connecting is less than $\varepsilon\delta_1^{-1}$.

(ii) For each γ with arc length at least $2\delta_1$ and $\gamma \subset Q$, there exists $G \in \mathscr{G}(Q, \delta_1, m)$ such that $\gamma \subset G$ and γ intersects properly G.

Remark. Statement (i) means that the total measure of not α-connecting squares has order $o(\delta_1)$.

4. METHOD OF MARKOV SIEVE

4.1. Preliminaries

Now we give the definition of the Markov sieve for general measure-preserving maps of compact metric spaces.

Definition 4.1. Let T be a map of a compact metric space $\mathscr{M}$ preserving the probability measure μ. A sequence of partitions $\mathscr{G}_{n,N} := \{A_0, A_1, ..., A_I\}$, where $I = I(n, N)$, $\mu(A_i \cap A_j) = 0$ for $i \neq j$, and $\mu(\mathscr{M} \setminus \bigcup_{i=0}^{I} A_i) = 0$, is called a *Markov sieve* with parameters $C_1 > 0$, $C_2 > 0$, and $1 > C_3 > 0$ if the following four properties hold:

Property G1 (Sizes). $\operatorname{diam}(A_i) \leqslant e^{-n}$ for $i = 1,..., I$ (A_0 is exceptional).

Property G2 (Measure of marginal set). $\mu(A_0) \leqslant Ne^{-n}$.

Property G3 (Markov approximation). For arbitrary natural numbers $k > l > 1$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$ and for arbitrary indices $j_1, j_2, ..., j_k \in \{1, ..., I\}$

$$\begin{aligned}\mu(T^{i_1}A_{j_1} \cap T^{i_2}A_{j_2} \cap \cdots \cap T^{i_{l-1}}A_{j_{l-1}} \mid T^{i_l}A_{j_l} \cap \cdots T^{i_k}A_{j_k}) \\ = \mu(T^{i_1}A_{j_1} \cap T^{i_2}A_{j_2} \cap \cdots \cap T^{i_{l-1}}A_{j_{l-1}} \mid T^{i_l}A_{j_l})(1+\delta)\end{aligned}$$

where $\delta \leqslant e^{-C_1 n}$. Set

$$R(i,k) := \{j \mid \mu(T^k A_i \cap A_j) \geqslant C_2 \mu(A_i)\,\mu(A_j)\}$$

Property G4 (Regularity). If $k \geqslant C_3 n$, then

$$\sum_{j \in R(i,k)} \mu(A_j) > 1 - e^{-n} \tag{3}$$

Remark. The last condition means that the mixing property for k steps provided by inequality (3) holds for "almost all" pairs A_i, $A_j \in \mathscr{G}_{n,N}$.

The following proposition on the convergence to the equilibrium is the analog of the classical Markov theorem and its proof can be carried out using properties G1–G4 only.

Proposition 4.2. Set $L=(i_{l+1}-i_l)/(2C_2n)$. For every $k>l>1$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N$ there exists a set $R_*(i_1,..., i_k)$ of $(k-l)$-tuples of indices and a constant $C_4>0$ such that

$$|\mu(T^{i_1}A_{j_1} \cap T^{i_2}A_{j_2} \cap \cdots \cap T^{i_{l-1}}A_{j_{l-1}} \mid T^{i_l}A_{j_l} \cap \cdots T^{i_k}A_{j_k})$$
$$-\mu(T^{i_1}A_{j_1} \cap T^{i_2}A_{j_2} \cap \cdots \cap T^{i_{l-1}}A_{j_{l-1}} \mid T^{i_l}A_{j_l})| < \delta_1$$

where $\delta_1 < \max(e^{-C_4 n}, (1-C_3)^L)$.

Remark. Usually $N>n$; if the typical value for $(i_{l+1}-i_l)$ is of order N, then the rate of the convergence to the equilibrium is determined by (N/n). The estimation for the decay of correlations can be derived from Proposition 4.2. Assume that a function $F(x)$, $x \in \mathscr{M}$, is Hölder continuous with exponent α and $\mathrm{E}F(x)=0$.

4.2. Decay of Correlations

Averaging $F(x)$ on the elements A_j of $\mathscr{G}_{n,N}$, we get a simple function $\tilde{F}(x)$, generating a stationary stochastic process: $\tilde{X}_k = \tilde{F}(T^k x)$ (of course $\tilde{X}_k$ is defined for $k \in \{1,..., N\}$ only). Properties G1 and G2 and the Hölder property of $F(x)$ imply

$$|\mathrm{E}F(x)\, F(T^N x) - \mathrm{E}\tilde{X}_0 \tilde{X}_N| \leqslant C(F)\, \gamma(\alpha)^n$$

$C(F)$ depends only on F, and $\gamma(\alpha)^n$ depends only on the exponent in the Hölder condition. Denote by f_i the value of $\tilde{F}(x)$ on $A_i \in \mathscr{G}_{n,N}$. Then

$$\mathrm{E}\tilde{X}_0 \tilde{X}_N = \sum_{i,j=0}^{I} f_i f_j \mu(A_i \cap A_j) = \sum_{i,j=0}^{I} f_i f_j \mu(A_i \mid A_j)\, \mu(A_j)$$

Proposition 4.2 implies that for $N>n^2$

$$\mathrm{E}\tilde{X}_0 \tilde{X}_N = \sum_{i,j=0}^{I} f_i f_j \mu(A_i)\, \mu(A_j) + O(\gamma_1^n) \tag{4}$$

for some $\gamma_1 < 1$. This last estimation proves the "conditional" theorem.

Theorem 4.3. If for a measure-preserving map T on a compact metric space $\mathcal{M}$ a Markov sieve $\mathcal{G}_{n,N}$ can be constructed with index rates $n^2 = N$, then every Hölder continuous function F obeys a stretched exponential correlation decay with the speed $C(F)\exp(-\sqrt{n})$, where the constant $C(F)$ depends only on F.

N. Chernov constructed a more effective Markov sieve for the PLH of the 2-torus, making it possible to prove exponential decay of correlations under the action of the map for every Hölder continuous function. Here we give Chernov's definition also in an abstract form, disregarding the nature of the sets of the sequence of the partitions entering the definition.

Definition 4.4. Let T be a piecewise linear hyperbolic map of the 2-torus. A sequence of partitions $\mathcal{G}_n := \{A_0, A_1, ..., A_I\}$, where $I = I(n)$, $\mu(A_i \cap A_j) = 0$ for $i \neq j$, and $\mu(\mathcal{M} \setminus \bigcup_{i=0}^{I} A_i) = 0$, is called a *Markov sieve* with parameters $C_1 > 0$ and $C_2 > 0$ if the following four properties hold:

Property G′1 (Sizes). $\operatorname{diam}(A_i) \leqslant e^{-n}$ for $i = 1, ..., I$ (A_0 is exceptional).

Property G′2 (Measure of marginal set). $\mu(A_0) \leqslant C_1 e^{-C_2 n}$.

Property G′3 (Exponential mixing for $A_i \in \mathcal{G}_{n,N}$):

$$\mu(T^k A_i \cap A_j) \geqslant \mu(A_i)\,\mu(A_j)(1 - C_1 e^{-C_2 n})$$

Property G′4 (Exponential mixing for a fixed parallelogram A). There exists a parallelogram $A \notin \mathcal{G}_n$ independent of n such that

$$\mu(T^k A \cap A) \geqslant \mu(A)^2 (1 - C_1 e^{-C_2 n})$$

Notice that the uniform bound in Property G′3 is a consequence of the existence of the parallelogram A and of the property G′4. Now property G′2 implies the inequality

$$\mathrm{E}\tilde{X}_0 \tilde{X}_n = \sum_{i,j=0}^{I} f_i f_j \mu(A_i)\,\mu(A_j) + O(\gamma_1^n)$$

analogous to inequality (4), but here we have $N = n$. Thus the following theorem is proven:

Theorem 4.5. Let T be a PLH of a 2-torus. Then for every Hölder continuous function, F obeys exponential correlation decay with speed $C(F)\,e^{-C_2 n}$, where C_2 depends only on T, while $C(F)$ depends on T and F.

4.3. The Idea of Constructing Markov Sieves

Definitions 4.1 and 4.4 contain no information on the nature of the elements $A_i \in \mathscr{G}_{n,N}$ (or $\in \mathscr{G}_n$) except their measurability. The Markov sieves for the corresponding systems are constructed by Bunimovich *et al.*[8,9] and Chernov.[10] The idea of constructing a Markov sieve via a Markov partition comes from the Adler and Weiss[11] construction of a Markov partition for the algebraic automorphism of the 2-torus consisting of two parallelograms A_1 and A_2 (see Fig. 3: A_1 is filled with o's, A_2 is filled with x's).

We suggest that the reader study this elementary geometrical picture after having read the general definitions of the Markov partition and the Markov sieve.

It is easy to check that the connected pieces of the refinements of the Adler–Weiss partition satisfy all requirements of both definitions of Markov sieves. In fact, using these partitions, one can encode the map T into the usual irreducible aperiodic Markov chain with finite state space. Hence the name "Markov partition."

While the abstract definition of Markov sieves is based on the existence of a metric and an invariant (under the action of T) measure defined on $\mathscr{M}$, the definition of Markov partition requires finer topological structure, namely the existence of local stable and unstable fibers. This definition

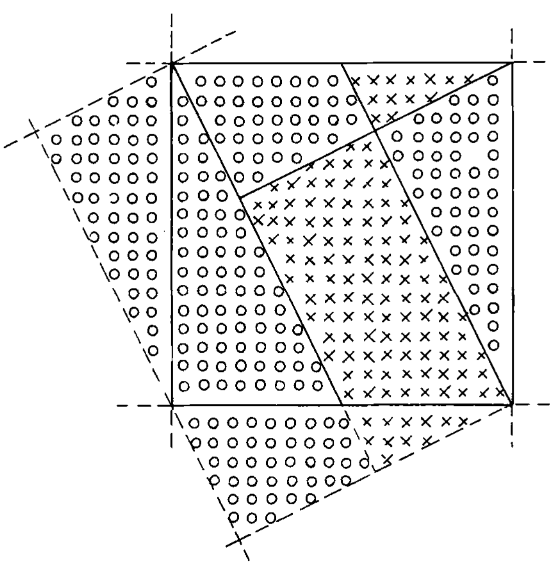

Fig. 3. The Adler–Weiss construction for $A = (2111)$.

has purely topological character, the main task here—after construction of a Markov parition (or a weakened variant of it—the so-called pre-Markov partition)—is to convert the topological conditions imposed on the Markov partition or pre-Markov partition into appropriate measure-theoretic properties. The first result in this direction was achieved by Sinai.[22,23] His construction for Anosov maps lies at the core of all constructions; therefore we shall summarize Sinai's idea.

Before doing this, we give the definitions of Markov partition and pre-Markov partition.

Definition 4.6 (*Markov partition*). A partition $\mathscr{G}$ of the phase space $\mathscr{M}$ into parallelograms of an Anosov map (or a UHS) is Markov iff it consists of a finite or countable number of parallelograms Q, such that for almost every $x \in \mathscr{M}$

$$T\gamma^s_{x,Q} \subseteq \gamma^s_{Tx,Q}$$

and

$$T^{-1}\gamma^u_{x,Q} \subseteq \gamma^u_{T^{-1}x,Q}$$

(For the notation $T\gamma^s_{x,Q}$ and $T\gamma^u_{x,Q}$ see Definition 3.4.)

Notice that if $\mathscr{G}$ is a Markov partition for T^n for some n, then the common refinement $\mathscr{G} \cup T\mathscr{G} \cup \cdots \cup T^{n-1}\mathscr{G}$ of the iterates of $\mathscr{G}$ is a Markov partition for T, too.

If one can construct Markov partitions whose elements are "full" parallelograms, then the Markovian condition can be formulated in a way that is easy to check.

Set

$$\partial^s\mathscr{G} = \bigcup_{A_i \in \mathscr{G}} \text{stable sides of } Q_i$$

$$\partial^u\mathscr{G} = \bigcup_{A_i \in \mathscr{G}} \text{unstable sides of } Q_i$$

The partition $\mathscr{G}$ of full parallelograms is Markov if and only if $T\partial^s\mathscr{G} \subseteq \partial^s\mathscr{G}$ $(T^{-1}\partial^u\mathscr{G} \subseteq \partial^u\mathscr{G})$.

Definition 4.7 (*Pre-Markov partition*). Let $\mathscr{G}$ be the parition of the phase space $\mathscr{M}$ of a singular hyperbolic (UHS, PLH, billiard) into polygons Q_i whose boundaries consist of local stable and local unstable fibers and eventually of singularity lines. As in the definition of Markov partition of full parallelograms, set

$$\partial^s \mathscr{G} = \bigcup_{Q_i \in \mathscr{G}} \text{stable sides of } Q_i$$

$$\partial^u \mathscr{G} = \bigcup_{Q_i \in \mathscr{G}} \text{unstable sides of } Q_i$$

(Notice that a polygon Q_i may have more than two stable (unstable) sides.) The partition $\mathscr{G}^{(n)}$ is *pre-Markov* of index n iff:

(i) The polygons nonadjacent to $\mathscr{S}_{-n,n}$ are quadrilaterals.

(ii) We have

$$T\partial^s \mathscr{G} \subseteq \partial^s \mathscr{G}$$

and

$$T^{-1}\partial^u \mathscr{G} \subseteq \partial^u \mathscr{G}$$

Observe that there is no condition imposed on the boundaries of Q_i being parts of $\mathscr{S}_{-\infty,\infty}$

4.4. Idea of Sinai's Construction

First take an initial partition $\mathscr{G}_0$ of $\mathscr{M}$ into a finite number of sufficiently small quadrilaterals Q_i whose sides are local stable and unstable fibers. A point $x \in \mathscr{M}$ is a double point of the boundary of the partition if it belongs to the intersection of a stable (unstable) side of a quadrilateral Q_i and an unstable (stable) side of another quadrilateral Q_j; see Fig. 4. We assume that the double points divide the boundaries of the quadrilaterals "regularly."

Using the notations of Fig. 4, this means that there exists a universal constant δ such that

$$\delta < \frac{\text{arc length}(a, x)}{\text{arc length}(x, b)} < \delta^{-1}$$

If n and therefore the expansion coefficient λ^n of T^n is large enough, then the regularity of $\mathscr{G}_0$ provides that the following construction is realizable. Take a stable side, say $\gamma_{a,b}$, of a $Q \in \mathscr{G}_0$ (see Fig. 4). $T^n\gamma_{a,b}$ will be a small piece of a stable fiber; project it by π^u into the closest stable side of $\mathscr{G}_0$, say $\gamma_{c,d}$ (if it is impossible, elongate $\gamma_{c,d}$ a bit, as in Fig. 4). Denote by $\gamma_{c',d'}$ this projection. Due to the contraction in the unstable direction of the map T^{-1}, $T^{-n}\gamma_{c',d'}$ will be λ^n times closer to $\gamma_{a,b}$ than $T^n\gamma_{a,b}$ will be

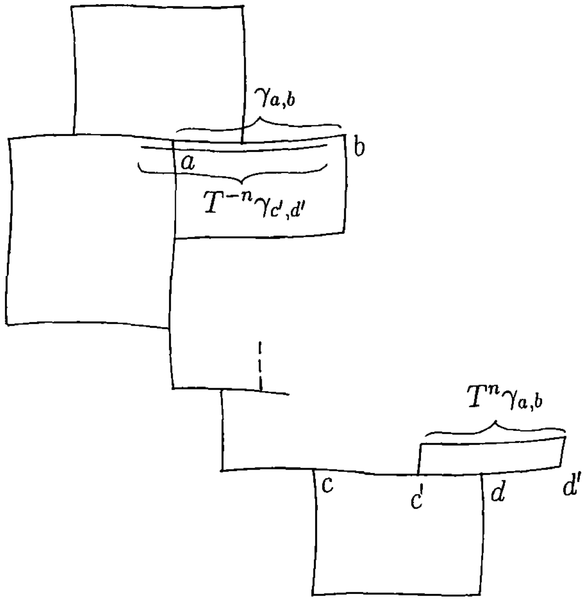

Fig. 4. The Sinai construction.

to $\gamma_{c',d'}$. Thus we can modify $\mathscr{G}_0$ in such a way that a piece of stable fiber having a great common part with $T^{-n}\gamma_{c',d'}$ replaces $\gamma_{a,b}$, but the structure of parallelograms Q_i, namely the regularity of double points, is preserved. Carrying out this procedure for all s-sides in $\mathscr{G}_0$, we get a partition G_1 for which the maximum distances of the type $T^n\gamma_{a,b}$ will be λ^n times closer to the closest stable side than it was in $\mathscr{G}_0$. Iterating this procedure in the limit, we get a regular partition G^s_∞ satisfying the Markov condition imposed on the s-sides of a partition consisting of full parallelograms. Repeating the above construction for the u-sides results in a Markov partition.

The above idea can be used for constructing pre-Markov partitions of index n for PLH and billiard maps. The effectiveness of Sinai's construction is enhanced by the fact that the starting partition for pre-Markov partitions consists of polygons having only approximate fibers as s-sides (u-sides). Nonetheless the iteration procedure transforms them into real local fibers.

Now we briefly summarize the idea of transforming a sequence of pre-Markov partitions into a Markov sieve.

Letting the index n tend to infinity and choosing the diameters of the polygons in $\mathscr{G}^{(n)}$ less than e^{-n}, one can achieve that the total measure of adjacent polygons is less than $\mu(A_0)\leqslant Ne^{-n}$ (for the billiard map) or less than $\mu(A_0)\leqslant C_1e^{-C_2n}$ (for the PLH map—here one should keep a quadrilateral of fixed size). The proof uses the structure of singularity lines.

Now set $A_i \in \mathscr{G}_n$, $i > 0$, the maximal parallelograms $A(Q_i)$ for nonadjacent quadrilaterals of $\mathscr{G}^{(n)}$. Since the expansion rate is not uniformly bounded from above, much more technical difficulties arise for proving that $\mu(\mathscr{M} \setminus \bigcup_{i>0} A_i)$ has the same order for both types of pre-Markov partitions.

5. METHOD OF HILBERT METRIC

This section is devoted to a summary of Liverani's recent paper[16] on the Hilbert metric method for proving correlation decay.

For this method the natural object is the action of the Perron–Frobenius operator on the density function $g(x)$ of μ [$g(x)$ is the Radon–Nikodym derivative of μ with respect to some dominating measure μ_0]. In general there is no simple formula for $\tilde{T}$, but if T is invertible and measure preserving, then $\tilde{T}$: $L^1_{\mu_0}(\mathscr{M}) \to L^1_{\mu_0}(\mathscr{M})$ is given by

$$\tilde{T}g(x) = g(T^{-1}x)$$

Recall that formula (1) (Section 1) describes the action of the Perron–Frobenius operator on the density function in the case of a one-dimensional (noninvertible and non-measure-preserving) map Tx, $x \in \mathbf{R}$:

$$\tilde{T}g(y) = \sum_{x \in T^{-1}y} g(T^{-1}y) \, |D_x T^{-1} x| \tag{5}$$

It is natural to expect that the two types of statements can be derived simultaneously by the same method for more general dynamical systems, too. Liverani carried out this program completely only for a special class of uniformly expanding maps of the unit interval. In this section we illustrate Liverani's method on the hyperbolic map of the unit interval, and we try to demonstrate how it works for Anosov maps and UHS maps. In both cases the existence of an invariant measure absolutely continuous with respect to the Riemannian are is assumed, together with the complete arsenal of the theory of hyperbolic systems (the mixing property and some standard constructions, including Sinai's lemma cited in Section 1). Now we turn to the definition of the Hilbert metric following Birkhoff.[6]

Consider a topological vector space $\mathbf{V}$ with a partial ordering $\preccurlyeq$ defined via a closed convex cone $\mathscr{C} \subset \mathbf{V}$ (u, $v \in \mathscr{C}$ implies that for all $\alpha > 0$, $\beta > 0$, $\alpha u + \beta v \in \mathscr{C}$; by convention $u = 0 \notin \mathscr{C}$):

$$f \preccurlyeq g \Leftrightarrow g - f \in \mathscr{C} \cup \{0\}$$

It is then possible to define a projective metric Θ (Hilbert metric) in $\mathscr{C}$ by the construction

$$\alpha(f, g) = \sup\{\lambda \in \mathbf{R}^+ \mid \lambda f \preccurlyeq g\}$$
$$\beta(f, g) = \inf\{\mu \in \mathbf{R}^+ \mid g \preccurlyeq \mu f\}$$
$$\Theta(f, g) = \log\left[\frac{\beta(f, g)}{\alpha(f, g)}\right]$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

Clearly, for all $\alpha > 0$, $\beta > 0$, $\Theta(\alpha f, \beta g) = \Theta(f, g)$ (see Fig. 5).

At the core of the proof of the correlation decay for any type of map lies the following "contraction principle."

Theorem 5.1. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two vector spaces and $T: \mathbf{V}_1 \to \mathbf{V}_2$ be a linear map such that $T(\mathscr{C}_1) \subset \mathscr{C}_2$ for two given closed convex cones $\mathscr{C}_1 \subset \mathbf{V}_1$ and $\mathscr{C}_2 \subset \mathbf{V}_2$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathscr{C}_i$. Setting

$$\Delta = \sup_{f, g \in T(\mathscr{C}_1)} \Theta_2(f, g)$$

we have

$$\Theta_2(Tf, Tg) \leqslant \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f, g), \qquad \forall f, g \in \mathscr{C}_1$$

$[\tanh(\infty) = 1]$.

Notice that if $T(\mathscr{C}_1) \subset \mathscr{C}_2$, then it follows that $\Theta_2(Tf, Tg) < \Theta_1(f, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.

Of course the contraction of the Hilbert metric itself does not ensure any convergence in the metric needed to estimate the rate of the correlation

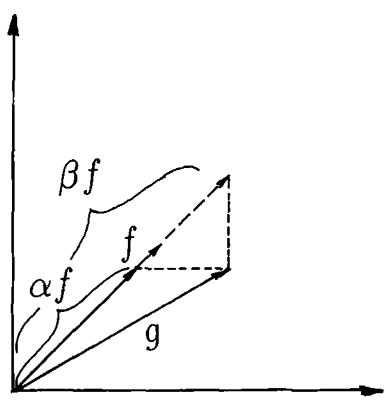

Fig. 5. The Hilbert metric.

decay. The following lemma is a candidate to solve this problem, but in the most interesting cases, the cone of functions considered is rather complicated, and the lemma is not applicable.

Lemma 5.2. Let $\|\cdot\|$ be a norm in $\mathbf{V}$, and suppose that, for each $f, g \in \mathbf{V}$,

$$-f \preccurlyeq g \preccurlyeq f \Rightarrow \|f\| \geqslant \|g\|$$

Then, given f, $g \in \mathscr{C} \subset V$ for which $\|f\| = \|g\|$,

$$\|f-g\| \leqslant (e^{\Theta(f,g)} - 1)\,\|f\|$$

We now illustrate the method.

5.1. The Simplest Uniformly Expanding Map of the Unit Interval

Let I_1, I_2 be two closed intervals such that $I_1 \cup I_2 = [0, 1]$, $I_1 \cap I_2 = \partial I_1 \cap \partial I_2$, and T_i: $I_i \to [0, 1]$, one to one and onto, such that $T_i \in C^{(2)}(I_i)$ and $DT_i \geqslant \lambda > 1$ $(i = 1, 2)$. Define the map as follows:

$$T(x) = \begin{cases} T_1(x) & \text{if} \quad x \in I_1 \\ T_2(x) & \text{if} \quad x \in I_2 \end{cases}$$

The Perron–Frobenius operator looks like (5); now the summation runs over the two branches of T. The appropriate cone of functions is defined by the relation

$$\mathscr{C}_a = \left\{ g \in C^{(0)} \mid \forall x, y \in [0, 1], g(x) > 0; \frac{g(x)}{g(y)} \leqslant e^{a|x-y|} \right\} \tag{6}$$

The functions $g \in \mathscr{C}_a$ obey a "logarithmic Lipschitz" condition with constant a. The Hilbert metric Θ associated to $\mathscr{C}_a$ looks as follows.
For each f, $g \in \mathscr{C}_a$

$$\Theta(f, g) = \ln \sup_{\substack{x, y \in [0,1] \\ u, v \in [0,1]}} \frac{(e^{a|x-y|}g(y) - g(x))(e^{a|u-v|}f(v) - f(u))}{(e^{a|x-y|}f(y) - f(x))(e^{a|u-v|}g(v) - g(u))}$$

The following two statements ensure the applicability of Theorem 5.1.

Statement A. Given $\sigma \in (\lambda^{-1}, 1)$, $\tilde{T}\mathscr{C}_a \subset \mathscr{C}_{\sigma a}$ provided $a \geqslant D/(\sigma - \lambda^{-1})$, where

$$D = \sup_{\substack{x \in [0,1] \\ i \in \{1,2\}}} \left| \frac{D_x^2 T_i^{-1}}{D_x T_i^{-1}} \right|$$

Statement B. We have

$$\Delta = \operatorname{diam}(\mathscr{C}_{\sigma a}) = 2 \ln \frac{1+\sigma}{1-\sigma} + 2\sigma a$$

Thus $\forall f, g \in \mathscr{C}_a$

$$\Theta(\tilde{T}f, \tilde{T}g) \leqslant \Lambda \Theta(f, g)$$

with

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right)$$

In this case Lemma 5.2 can be applied, because for each g_1, $g_2 \in C^{(0)}([0, 1])$, $-g_1 \preccurlyeq g_2 \preccurlyeq g_1$ implies $-g_1(x) \leqslant g_2(x) \leqslant g_1(x)$, that is, $|g_2(x)| \leqslant g_1(x)$, for each $x \in [0, 1]$. Consequently, for each L^p norm, $\|g_2\|_p \leqslant \|g_1\|_p$.

Let $g \in C^{(0)}([0, 1])$, $\int_0^1 g = 1$, and $g \in \mathscr{C}_{a_*}$, where a_* and σ_* satisfy the conditions of Statement A.

Since $\int_0^1 \tilde{T}^n g = \int_0^1 g = 1$,

$$\begin{aligned} \|\tilde{T}^{n+m}g - \tilde{T}^n g\|_1 &\leqslant \exp[\Theta(\tilde{T}^{n+m}g, \tilde{T}^n g)] - 1 \\ &\leqslant \exp[\Lambda^{n-1}\Theta(\tilde{T}(\tilde{T}^m g), \tilde{T}g)] - 1 \leqslant [\exp(\Delta\Lambda^{-1})]\, \Delta\Lambda^{n-1} \end{aligned}$$

This means that $\{\tilde{T}^n g\}$ is a Cauchy sequence in L^1; in addition, $\{\tilde{T}^n g\}$ are equicontinuous; thus the limit $\varphi_* = \lim_{n \to \infty} \tilde{T}^n g$ exists and φ_* is the density of the unique invariant measure absolutely continuous with respect to the Lebesgue measure.

The following theorem—being the consequence of the preceding considerations—is slightly more general than the exponential correlation decay for smooth functions.

Theorem 5.3. There exist K, $r \in \mathbf{R}^+$ such that for each $f \in L^1([0, 1])$, $g \in C^{(1)}([0, 1])$ with $\int_0^1 g = 1$

$$\left| \int_0^1 f(T^n(g(x)))\, dx - \int_0^1 f(x)\, \varphi_*(x)\, dx \right| \leqslant K \|f\|_1 (\|g\|_1 + r\, \|g'\|_\infty)\, \Lambda^n$$

5.2. The Anosov Map

For the definition see Section 2. Here we add only one construction to it, which can be done due to the uniform transversality of stable and unstable vectors.

There exists a bundle of cones $\{\mathscr{C}(x)\}_{x\in\mathscr{M}}$ in the tangent bundle [i.e., $\mathscr{C}(x)\subset\mathscr{T}_x\mathscr{M}$] strictly invariant [i.e., $D_xT^{-1}\mathscr{C}(x)\subset \operatorname{int}\mathscr{C}(T^{-1}x)\cup\{0\}$] and continuous. $\mathscr{C}(x)$ can be a domain between two straight lines in $\mathscr{T}_x\mathscr{M}$ defined by the vectors $[v^u(x)+v^s(x)]/2$ and $[v^s(x)+v^u(x)]/2$; here we assume that $\|v^u(x)\|=\|v^s(x)\|=1$ and the expanding eigenvalue $\lambda(x)>0$. We can suppose that

$$\inf_{v\in\mathscr{C}(x)} \|D_xT^{-1}v\|\geqslant\lambda$$

$$\sup_{\substack{v,w\in\mathscr{C}(x)\\ \|v\|=\|w\|=1}} \|v-w\|\leqslant\tfrac{1}{2}$$

maybe for a λ less than that in Definition 3.1.

The appropriate cone of functions will be defined by a logarithmic Lipschitz condition similar to (6) imposed on integral means of test functions defined on expanding curves. The tangent vectors to these auxiliary curves $\gamma\in\Gamma_\delta$ at every x belong to the expanding cone $\mathscr{C}(x)$ and have bounded curvature:

$$\Gamma_\delta=\{\gamma\subset\mathscr{M}\mid \delta\leqslant \operatorname{length}(\gamma)\leqslant 2\delta;\ \forall x\in\gamma,\ \gamma'(x)\subset\mathscr{C}(x);\ |\kappa_\gamma(x)|\leqslant H\}$$

The test functions defined on $\gamma\in\Gamma_\delta$ obey a logarithmic Hölder condition with exponent $\nu\in(0,1/2]$; they form a cone, too:

$$\mathscr{D}_a(\gamma)=\left\{f\in C^{(0)}(\gamma)\mid f>0;\frac{f(x)}{f(y)}\leqslant e^{ad(x,y)^\nu}\right\} \tag{7}$$

where d means the arc length distance along γ. The explicit formula for the Hilbert metric $\rho_\gamma(f_1,f_2)$ on $\mathscr{D}_a(\gamma)$ can be written, as for $\Theta(f,g)$. Focusing on the main ideas only, we omit its exact form.

Now we are in the position to introduce the cone of functions for which the contraction principle will be applied:

$$\mathscr{C}_{b,c}=\left\{g\in C^{(0)}(\mathscr{M})\,\middle|\,\forall\gamma\in\Gamma_\delta,\ \forall x\in\gamma,\ \forall f,f_1,f_2\in\mathscr{D}_a(\gamma)\int_\gamma gf>0;\right.$$
$$\left.\frac{f_2(x)\int_\gamma gf_1}{f_1(x)\int_\gamma gf_2}\leqslant e^{b\rho_\gamma(f_1,f_2)};\ \|D^ug\|_\infty\leqslant c\inf_{\substack{\gamma\in\Gamma_\delta\\ f\in\mathscr{D}_a(\gamma)}}\frac{\int_\gamma fg}{\int_\gamma f}\right\}$$

By D^u we mean the derivative in the unstable direction.

Due to condition (7), the inequality $\int_\gamma gf > 0$ does not imply the nonnegativity of g: the Dirac delta function cannot be approximated with arbitrary accuracy by functions $f \in \mathscr{D}_a(\gamma)$. The second inequality in the definition of the cone $\mathscr{C}_{b,c}$ is the "weak" variant of the second inequality in the definition (6) of $\mathscr{C}_a$.

The following fundamental lemma ensures the application of the contraction principle.

Lemma 5.4. Let $\delta \leqslant \delta_0$; then there exist $N_* \in \mathbf{N}$ and $\Delta \in \mathbf{R}^+$ such that

$$\text{diameter}(\tilde{T}^{N_*}\mathscr{C}_{b,c}) \leqslant \Delta < \infty$$

The proof of this lemma uses the mixing property of T, which provides that for every $\gamma \in \Gamma_\delta$ there exists a natural N such that $T^N\gamma$ has a subinterval γ_1 sufficiently close to γ. The necessary estimates are based on the absolute continuity of the canonical projection π^s between γ and γ_1. In order to get the desired estimate, we should assume that the Γ_δ is a family of smoothly varying curves γ.

Lemma 5.2 does not apply to the Hilbert metric defined by the cone $\mathscr{C}_{b,c}$; therefore some extra effort is needed to prove the following result.

Theorem 5.5. There exist $K > 0$ and $r > 0$ such that, for each $g, f \in C^{(1)}(\mathscr{M})$,

$$\left| \int_{\mathscr{M}} f\tilde{T}^n g - \int_{\mathscr{M}} f \int_{\mathscr{M}} g \right| \leqslant K \|g\|_* \|f\|_* \Lambda^n$$

with $\|h\|_* = \int_{\mathscr{M}} |h| + r\|h'\|_\infty$.

The result can be obtained using Markov partitions, but Liverani overcomes this difficulty by construcing two series of quasipartitions obeying much milder conditions than Markov ones.

5.3. The UHS Map

The UHS map T was also defined in Section 2. We suggest that the reader keep in mind one of the examples for Chernov's PLH map. The contracting—under the action of $\tilde{T}$—cone $\mathscr{C}_{\text{singular}}$ of functions belonging to

$$C^{(0)}\left(\mathscr{M} \setminus \bigcup_{n=0}^{\infty} T^n \mathscr{S}^-\right) \cap L^1(\mathscr{M})$$

will be defined analogously to $\mathscr{C}_{b,c}$.

Now the map T can break up curves $\gamma \in \Gamma_\delta$ into arbitrarily small pieces; therefore the definition of the cone should maintain short curves, too. This requires the introduction of another triplet of test functions, defined on curves $I \in \Gamma_{|I|}$, $|I| < \delta$. We do not burden the reader with a complicated formula defining $\mathscr{C}_{\text{singular}}$; we focus attention on a crucial point only: one of the tree extra conditions is of the type

$$\frac{|\int_I g f_3|}{f_3(y)} \leqslant \text{const} \cdot d\delta^{1-\zeta} \, |I|^\zeta$$

where const depends only on g, d is a new parameter of the cone, and $0 < \zeta < 1$ is the exponent in (2). This means that an integral can be estimated by some power smaller than 1 of the length $|\gamma_I|$ of its domain of integration, when $|\gamma_I|$ is arbitrarily small. The weakness of the condition imposed on $g \in \mathscr{C}_{\text{singular}}$ enables Liverani to control the speed of the mixing for the squares entering Sinai's fundamental lemma, and to prove the exponential correlation decay for each g, $f \in C^{(1)}(\mathscr{M})$ as in Theorem 5.5. The main steps are the same as in the proof of the smooth case, but the estimates are much more complicated.

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